A New Statistical Max Operation for Propagating Skewness in Statistical Timing Analysis

Kaviraj Chopra Bo Zhai David Blaauw Dennis Sylvester
Department of EECS
University of Michigan
Ann Arbor,MI

ABSTRACT

Statistical static timing analysis (SSTA) is emerging as a solution for predicting the timing characteristics of digital circuits under process variability. For computing the statistical max of two arrival time probability distributions, existing analytical SSTA approaches use the results given by Clark in [8]. These analytical results are exact when the two operand arrival time distributions have jointly Gaussian distributions. Due to the nonlinear max operation, arrival time distributions are typically skewed. Furthermore, nonlinear dependence of gate delays and non-gaussian process parameters also make the arrival time distributions asymmetric. Therefore, for computing the max accurately, a new approach is required that accounts for the inherent skewness in arrival time distributions. In this work, we present analytical solution for computing the statistical max operation. First, the skewness in arrival time distribution is modeled by matching its first three moments to a so-called skewed normal distribution. Then by extending Clark's work to handle skewed normal distributions we derive analytical expressions for computing the moments of the max. We then show using initial simulations results that using a three moment based max operation can significantly improve the accuracy of the statistical max operation while retaining its computational efficiency.

1. INTRODUCTION

Process control precision is worsening with continuous process scaling due to smaller dimensions, smaller number of doping atoms and aggressive lithographic techniques. This results in an increase in process parameters fluctuations, that causes variations in electrical characteristics of transistors and interconnects. These variations in electrical characteristics of circuit components affect timing and result in chip operating frequency variation. Traditionally corner based static timing analysis have been used to guard against yield loss resulting from these variations; however, with increasing number of sources of variation, corner based methods are becoming overly pessimistic and computationally expensive.

An alternative approach, namely, statistical static timing analysis (SSTA) has emerged as a possible solution for statistically quantifying the variability in timing performance. Existing SSTA approaches can be broadly classified into block based SSTA [13, 4, 1, 9, 6, 17] and path based SSTA [14, 2]. A path-based SSTA requires enumeration of an exponential number of paths, therefore, block-based SSTA is considered to be a more efficient technique. Among these, the analytical [1] methods, presented in [3, 6, 17], appeal to be the more promising approaches for a computationally efficient implementation of SSTA. In [3], the author introduced a linear time analytical SSTA algorithm assuming uncorrelated normal random variables for delay distribution. Using a first order parametric delay model, a method for handling correlations in global sources of variation due to both spatial correlation and path reconvergence was presented in [6, 17]. Their SSTA algorithm included a PERT-like topological traversal of a circuit graph, where at each node the maximum arrival time distribution is computed in terms of the parametric delay model. For propagating arrival time distributions, one needs to compute the sum and the maximum of two arrival time at each node in the circuit graph. The computation of the sum function is relatively simple; however, the statistical max of two correlated arrival time variables is non-trivial.

The max operation in existing SSTA approaches is invariably based on analytical results given in [8]. Clark derived analytical expressions for finding the moments of the max of two correlated normal random variables and an expression for computing the correlation of the resulting max with any other jointly normal variable. The Clark's max results are exact when the two operand random variables have a bivariate normal distribution. However, the result of the max of two normal variables is typically a positively skewed non normal distribution. Skewness is a statistical parameter used to describe asymmetry in a random variables probability distribution. A probability distribution is said to have positive(negative) skewness if it has a long tail in the positive(negative) direction (see Figure 1). Both the above mentioned analytical approaches [6, 17], use these expressions for computing the moments of statistical max of two arrival time random variables. Unfortunately in SSTA, the asymmetric non-normal arrival time distributions resulting from the max operation performed at one node are inputs to the max operation which is needed to be performed at a downstream node. Additionally, variations in interconnect and few process parameters also have asymmetric non-normal distributions [19]. However, existing analytical SSTA approaches have to approximate the non-normal arrival time distribution with a normal distribution for applying Clark's max. The error of this approximation increases when the difference of the mean relative to the standard deviation decreases and it becomes maximum when two means are equal [8]. For a typical design, there can be several thousand critical paths and the

[1]As the focus of the paper is on block based analytical approaches here we limit discussion to relevant previous work on block based analytical SSTA methods.

means of their output arrival time distributions and arrival time distributions at common internal nodes will be closely aligned with each other. Therefore, in such a case Clark's max based SSTA methods may result in inadequate accuracy, in particular, for power optimized designs having a large number of nodes with zero or small slack. Recently, SSTA algorithms using higher order nonlinear parametric delay models with non gaussian distributions were proposed in [7, 12, 18, 19]. However, for computing the max operation, these approaches either use numerical techniques and/or employ the Clark's max requiring normal approximation. A conditional max based heuristic analytical method was presented in [19] where the max operations is postponed until the two arrival time distributions are skewed.

In this work, we extend Clark's max approach and give analytical results for computing an improved approximation for the maximum of a set of asymmetric random variables by preserving the first three moments. The problem of computing the max of a finite set of random variables has been well studied. Several approaches derived Clark's results using different methods [11, 5]. In our method, given the first three moments of any asymmetric distribution, we give analytical expressions to map it to a skewed normal (explained later) representation having same moments. We then derive analytical results for computing the moments of the max of two correlated skewed normal distributions. The derivation is similar in spirit to Clark's approach, although it is more general since we can compute the moments for a bivariate skewed normal random variables.

The rest of the paper is organized as follows. In Section 2 we explain the skewed normal distribution and give analytical expressions for computing the parameters of a skewed normal distribution from the mean, variance and skewness of arrival time distribution. A bivariate skewed normal distribution and the derivation for the proposed max operation are given in Section 3. In Section 4, we give numerical results illustrating the efficacy of the proposed max operation. Section 5 concludes the paper.

2. MODELING SKEWNESS

Arrival time distributions and circuit delay distributions are typically positively skewed, due to the nonlinear max operation and nonlinear dependence of delay on process parameters. We need an analytical representation that is flexible enough to capture the skewness in asymmetric arrival time distributions and at the same time be of the functional form which allows analytical derivation of the statistical max operation. After studying several skewed representations, in [10], we found a general method for introducing skewness into any unimodal symmetric distribution. Their basic idea is to simply introduce inverse scale factors in the left and the right half real lines around the mean. Let $f(x)$ be the normal distribution with mean μ and variance σ given by

$$f(x) = \frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma}), \text{ where } \phi(x) = \frac{1}{\sqrt{2\,\pi}}e^{-\frac{x^2}{2}}.$$

Using the method presented in [10], a skewed normal distribution $f_\gamma(x)$ can be computed from the normal distribution $f(x)$, by scaling its left half and right half by factors γ and its inverse $1/\gamma$, respectively. This gives us the skewed normal distribution,

$$\begin{aligned} f_\gamma(x) \;&=\; \frac{2}{\sigma\,(\gamma+1/\gamma)}\left\{\phi(\frac{(x-\mu)\gamma}{\sigma})I_{(-\infty,\mu]}(x)\right.\\ &\left.+\;\phi(\frac{x-\mu}{\gamma\sigma})I_{(\mu,\infty)}(x)\right\}, \end{aligned}$$

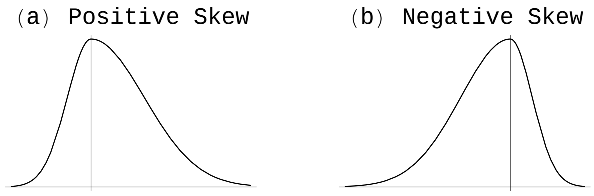

Figure 1: Examples for Asymmetric PDFs

where, $I_{(-\infty,\mu]}(x)$ and $I_{(\mu,\infty)}(x)$ are the Indicator functions:

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

For a skewed normal distribution, we can observe that scaling variable x corresponds to an inverse scaling of the standard deviation σ around its mean. Therefore, $f_\gamma(x)$ can be alternatively written as

$$\begin{aligned} f_\gamma(x) \;&=\; \frac{2}{\sigma_l+\sigma_r}\left\{\phi(\frac{(x-\mu)}{\sigma_l})I_{(-\infty,\mu]}(x)\right.\\ &\left.+\;\phi(\frac{x-\mu}{\sigma_r})I_{(\mu,\infty)}(x)\right\}, \end{aligned}$$

where,

$$\sigma_l = \tfrac{\sigma}{\gamma} \quad \text{and} \quad \sigma_r = \sigma\gamma.$$

Note that the resulting skewed distribution $f_\gamma(x)$ has a functional form similar to the original non-skewed distribution $f(x)$. If the skewness parameter γ is greater(less) than unity then $f_\gamma(x)$ is positively(negatively) skewed. For $\gamma = 1$ we get back the original symmetric normal distribution. Furthermore, $f_\gamma(x)$ is both continuous and differentiable and is completely defined by only three parameters μ, σ and γ. These were the key appealing properties that motivated us to use this representation for deriving the proposed max operation.

Existing SSTA approaches model and propagate only the mean and variance of the arrival time distribution. For improving the accuracy of SSTA algorithm, in addition to the mean and variance, we wish to propagate the skewness in asymmetric arrival time distributions. In such an SSTA framework, the input parameters of the max operation will include mean, variance and skewness of the two input arrival time distributions and their correlation. We first want to map the arrival time distribution characterized by its mean, variance and skewness to a skewed normal distribution $f_\gamma(x)$. Let μ_γ, σ_γ and Sk_γ be the given mean, standard deviation and skewness of a skewed arrival time distribution and μ, σ and γ are the three parameters that define the desired skewed normal distribution $f_\gamma(x)$. For finding $f_\gamma(x)$, we express the mean, variance and skewness of the skewed normal distribution as function of its parameters μ, σ and γ and then match these to the μ_γ , σ_γ and Sk_γ of a skewed arrival time distribution to solve for μ, σ and γ. The analytical expressions for mean μ_γ, variance σ_γ^2 and skewness Sk_γ of $f_\gamma(x)$ derived in terms of its parameters (μ, σ and γ) are given as follows:

$$\mu_\gamma = \mu + \sqrt{\frac{2}{\pi}}\,\left(\gamma - \frac{1}{\gamma}\right)\,\sigma \tag{1}$$

$$\sigma_\gamma^2 = \frac{\left(\pi\,\gamma^4 - 2\,\gamma^4 - \pi\,\gamma^2 + 4\,\gamma^2 + \pi - 2\right)\,\sigma^2}{\pi\,\gamma^2} \tag{2}$$

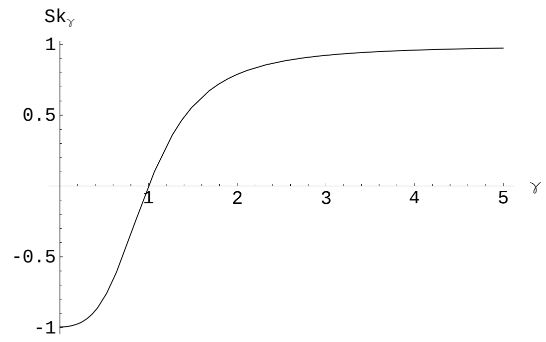

Figure 2: The γ parameter of $f_\gamma(x)$ vs. Skewness Sk_γ

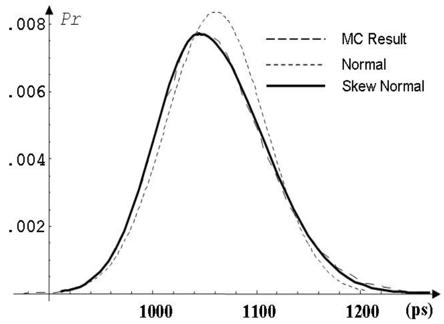

Figure 3: Comparison between Skewed Normal distribution and Normal distribution for a typical Monte Carlos based Arrival Time distribution.

The skewness of distribution defined by the ratio of the third centered moment and cubed standard deviation is given by

$$Sk_\gamma = \frac{\sqrt{2}\,\left(1-\gamma^2\right)\,\left(\pi\,\left(\gamma^4-3\,\gamma^2+1\right)-4\left(\gamma^2-1\right)^2\right)}{\left(\pi\,\left(\gamma^4-\gamma^2+1\right)-2\left(\gamma^2-1\right)^2\right)^{\frac{3}{2}}}. \tag{3}$$

Fortunately, the skewness Sk_γ (Eq. 3) is only a function of γ and is independent of the other two parameters μ and σ. A plot of this function is given in Figure 2, where it can be seen that skewness Sk_γ is a well behaved function and it monotonically increases with γ. Therefore, for a given Sk_γ, one can efficiently compute γ either using pre-computed look-up tables or using numerical methods with very fast convergence. Using γ, σ_γ and μ_γ we can analytically solve equations 2 and 1 for parameters σ and μ, respectively. Thus given mean, variance and skewness of an arrival time distribution we can easily map it to a skewed normal distribution. In Figure 3, we show plots of a typical skewed arrival time distribution approximated by a skewed normal distribution and normal distribution. It is evident that compared to existing normal approximations, skewed normal is a much better representation that can accurately capture the inherent skewness in arrival time distributions.

3. SKEWED NORMAL MAX OPERATION

Based on the skewed normal representation explained in the previous section, we now present the skewed normal max operation. For analytically expressing the max function of two correlated arrival time random variables X and Y, we need to know their joint probability distribution function. In [8], the author uses the following bivariate normal distribution for the two operand random variables.

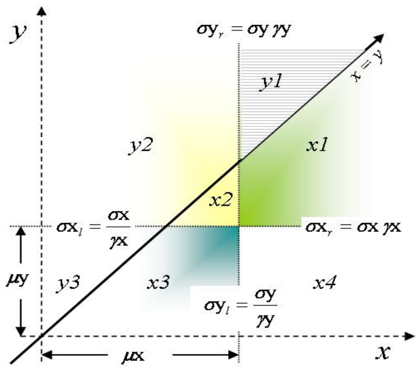

Figure 4: Standard deviations of a bivariate Skewed Normal distribution and seven regions of integration for μx $>$ μy

$$\begin{aligned} f(x,y) &= \frac{1}{2\pi\sigma\mathrm{x}\sigma\mathrm{y}}\phi\left(\frac{x-\mu\mathrm{x}}{\sigma\mathrm{x}}, \frac{y-\mu\mathrm{y}}{\sigma\mathrm{y}}\right) \\ & \qquad \text{where, } \phi(x,y) = \frac{1}{\sqrt{1-\rho^2}} e^{-\frac{x^2-2\rho xy+y^2}{2\left(1-\rho^2\right)}} \end{aligned}$$

Recall that bivariate normal representation being symmetric will introduce errors in the computation of the recursive max operation for SSTA purposes. Therefore, similar to the univariate skewed normal presented in the previous section, we add two inverse scale parameters γx and γy for random variables X and Y around their respective means μx and μy for introducing skewness

$$\begin{aligned} f_\gamma(x,y) &= \frac{1}{\Gamma\sigma\mathrm{x}\sigma\mathrm{y}}\Bigg(\\ & \quad \phi\left(\frac{x-\mu\mathrm{x}}{\sigma\mathrm{x}_l}, \frac{y-\mu\mathrm{y}}{\sigma\mathrm{y}_l}\right) I_{(-\infty,\mu\mathrm{x})}(x) I_{(-\infty,\mu\mathrm{y})}(y) \\ & + \phi\left(\frac{x-\mu\mathrm{x}}{\sigma\mathrm{x}_l}, \frac{y-\mu\mathrm{y}}{\sigma\mathrm{y}_r}\right) I_{(-\infty,\mu\mathrm{x})}(x) I_{[\mu\mathrm{y},\infty)}(y) \\ & + \phi\left(\frac{x-\mu\mathrm{x}}{\sigma\mathrm{x}_r}, \frac{y-\mu\mathrm{y}}{\sigma\mathrm{y}_l}\right) I_{\left[\mu_x,\infty\right)}(x) I_{(-\infty,\mu\mathrm{y})}(y) \\ & + \phi\left(\frac{x-\mu\mathrm{x}}{\sigma\mathrm{x}_r}, \frac{y-\mu\mathrm{y}}{\sigma\mathrm{y}_r}\right) I_{\left[\mu_x,\infty\right)}(x) I_{[\mu\mathrm{y},\infty)}(y)\Bigg) \\ \text{where } \Gamma &= \frac{\pi}{2}\left(\gamma\mathrm{x}+\frac{1}{\gamma\mathrm{x}}\right)\left(\gamma\mathrm{y}+\frac{1}{\gamma\mathrm{y}}\right) \\ & + \left(\gamma\mathrm{x}-\frac{1}{\gamma\mathrm{x}}\right)\left(\gamma\mathrm{y}-\frac{1}{\gamma\mathrm{y}}\right)\tan^{-1}\left(\frac{\rho}{\sqrt{1-\rho^2}}\right); \end{aligned}$$

$$\sigma\mathrm{x}_l = \tfrac{\sigma\mathrm{x}}{\gamma\mathrm{x}}; \quad \sigma\mathrm{y}_l = \tfrac{\sigma\mathrm{y}}{\gamma\mathrm{x}}; \quad \sigma\mathrm{x}_r = \sigma\mathrm{x}\ \gamma\mathrm{x}; \quad \text{and } \sigma\mathrm{y}_r = \sigma\mathrm{y}\ \gamma\mathrm{y}.$$

Due to the correlation ρ, the normalizing constant term Γ differs from the univariate case. We use this bivariate skewed normal distribution as a better approximation for modeling the jointly skewed arrival time distributions. Figure 4 graphically illustrates how the two indicator functions partition the X, Y plane into 4 quadrants having different standard deviations around the mean vector (μx, μx). Let $v(i)$ be the i^{th} moment of max(X, Y) given by

$$
\begin{aligned}
v(i) &= \int_{-\infty}^{\infty}\int_{\infty}^{\infty} (\max(x,y))^i f_\gamma(x,y)\, dydx \\
&= \oint\limits_{(x,y)\in X>Y} x^i f_\gamma(x,y)\, d(x,y) \\
&+ \oint\limits_{(x,y)\in X\leq Y} y^i f_\gamma(x,y)\, d(x,y)
\end{aligned}
$$

As shown in Figure 4 the region $X > Y$ gets further partitioned into 4 sub-regions x_1, x_2, x_3 and x_4 where the sub-script denotes the standard deviation quadrant and likewise, region $X \leq Y$ gets partitioned into sub-regions y_1, y_2 and y_3. Therefore, we can write the i^{th} moment of max(X, Y) as

$$v(i) = \sum_{j=1}^{3} v_{y,j}(i) + \sum_{j=1}^{4} v_{x,j}(i)$$

where, $v_{x,j}(i)$)and $v_{y,j}(i)$ are the i^{th} moment of max(x, y) in the j^{th} quadrant. The complete derivation of $v(i)$ over all the seven regions is too long and tedious. Therefore, in this paper we will present the key steps encountered while deriving the expression for moments of sub-region y_1. The i^{th} moment of max, for $Y > X$ in the 1[st] quadrant, is given as follows:

$$v_{y,1}(i) = \frac{1}{\Gamma\sigma\mathrm{x}\sigma\mathrm{y}} \int_{\mu\mathrm{x}}^{\infty}\int_{x}^{\infty} y^i \phi\left(\frac{x-\mu\mathrm{x}}{\sigma\mathrm{x}_r}, \frac{y-\mu\mathrm{y}}{\sigma\mathrm{y}_r}\right)\, dydx$$

Using the Lebnitz rule, we compute the partial derivative of $v_{y,1}(i)$ with respect to μx:

$$
\begin{aligned}
\frac{\partial v_{y,1}(i)}{\partial\mu\mathrm{x}} =& \frac{1}{\Gamma\sigma\mathrm{x}\sigma\mathrm{y}} \int_{\mu\mathrm{x}}^{\infty}\int_{x}^{\infty} y^i \frac{\partial}{\partial\mu\mathrm{x}}\phi\left(\frac{x-\mu\mathrm{x}}{\sigma\mathrm{x}_r}, \frac{y-\mu\mathrm{y}}{\sigma\mathrm{y}_r}\right)\, dydx \\
& - \frac{1}{\Gamma\sigma\mathrm{x}\sigma\mathrm{y}\sqrt{1-\rho^2}} \int_{\mu\mathrm{x}}^{\infty} y^i e^{-\frac{\left(\frac{y-\mu\mathrm{y}}{\sigma\mathrm{y}_r}\right)^2}{2\left(1-\rho^2\right)}}\, dy
\end{aligned}
$$

We first change order of integration variables in the inner integral,

$$
\begin{aligned}
\frac{\partial v_{y,1}(i)}{\partial\mu\mathrm{x}} =& \frac{1}{\Gamma\sigma\mathrm{x}\sigma\mathrm{y}\sqrt{1-\rho^2}} \int_{\mu\mathrm{x}}^{\infty} y^i e^{-\frac{(y-\mu y)^2}{2\sigma\mathrm{y}_r^2}} \\
& \int_{\mu\mathrm{x}}^{y} \frac{\partial e^{-\frac{\left(\frac{x-\mu\mathrm{x}}{\sigma\mathrm{x}_r} - \frac{\rho(y-\mu\mathrm{y})}{\sigma\mathrm{y}_r}\right)^2}{2\left(1-\rho^2\right)}}}{\partial\mu\mathrm{x}} dxdy \\
& - \frac{\gamma_1}{\Gamma\sigma\mathrm{x}\sigma\mathrm{y}\sqrt{1-\rho^2}} \int_{\mu\mathrm{x}}^{\infty} y^i e^{-\frac{\left(\frac{y-\mu\mathrm{y}}{\sigma\mathrm{y}_r}\right)^2}{2\left(1-\rho^2\right)}}\, dy
\end{aligned}
$$

Now, the inner integral of the first term in the above expression is in an integrable form. We evaluate this integral and an additional term due to the integration cancels out the second term and gives us the following simplified result.

$$
\begin{aligned}
\frac{\partial v_{y,1}(i)}{\partial\mu\mathrm{x}} =& -\frac{1}{\Gamma\sigma\mathrm{x}\sigma\mathrm{y}\sqrt{1-\rho^2}} \int_{\mu\mathrm{x}}^{\infty} y^i e^{-\frac{(y-\mu y)^2}{2\sigma\mathrm{y}_r^2}} \\
& e^{-\frac{\left(\frac{y-\mu\mathrm{x}}{\sigma\mathrm{x}_r} - \frac{\rho(y-\mu\mathrm{y})}{\sigma\mathrm{y}_r}\right)^2}{2\left(1-\rho^2\right)}}\, dy
\end{aligned}
$$

Similar to [8], we first make the substitution $y = \frac{\left(\sigma\mathrm{x}_r\sigma\mathrm{y}_r\sqrt{1-\rho^2}\right)n}{a} + \mu\mathrm{y} + \frac{(\mu\mathrm{x}-\mu\mathrm{y})\sigma\mathrm{y}_r(\sigma\mathrm{y}_r - \sigma\mathrm{x}_r\rho)}{a^2}$ and then $\mu\mathrm{x} = \mu\mathrm{y} + am$.

$$
\begin{aligned}
\frac{\partial v_{y,1}(i)}{\partial m} =& -\frac{1}{\Gamma} e^{-\frac{m^2}{2}} \int_{\frac{m(\sigma\mathrm{x}_r - \sigma\mathrm{y}_r\rho)}{\sigma\mathrm{y}_r\sqrt{1-\rho^2}}}^{\infty} e^{-\frac{n^2}{2}} \\
& \left(\mu\mathrm{y} + \frac{n\sigma\mathrm{x}_r\sigma\mathrm{y}_r\sqrt{1-\rho^2}}{a} + \frac{m\sigma\mathrm{y}_r\left(\sigma\mathrm{y}_r - \sigma\mathrm{x}_r\rho\right)}{a}\right)^i dn
\end{aligned}
$$

where,

$$a^2 = \sigma\mathrm{x}_r^2 + \sigma\mathrm{y}_r - 2\rho\sigma\mathrm{x}_r\sigma\mathrm{y}_r$$

Now note that for $m = \infty$, the random variable $X >> Y$ and therefore, at $m = \infty$ all moments $v_{y,1}(i) = 0$. Using this observation one can express $v_{y,1}(i)$ as follows:

$$v_{y,1}(i) = \frac{1}{\Gamma}\int_{\alpha}^{\infty} e^{-\frac{m^2}{2}} \int_{k_3 m}^{\infty} \left(\mu\mathrm{y} + k_1 n + k_2 m\right)^i e^{-\frac{n^2}{2}}\, dndm \qquad (4)$$

where,

$$
k_1 = \frac{\sigma\mathrm{x}_r\sigma\mathrm{y}_r\sqrt{1-\rho^2}}{a}, \qquad k_2 = \frac{\sigma\mathrm{y}_r\left(\sigma\mathrm{y}_r - \sigma\mathrm{x}_r\rho\right)}{a},
$$
$$
k_3 = \frac{\sigma\mathrm{x}_r - \sigma\mathrm{y}_r\rho}{\sigma\mathrm{y}_r\sqrt{1-\rho^2}} \quad \text{and} \quad \alpha = \frac{(\mu\mathrm{x} - \mu\mathrm{y})}{a}.
$$

For a given positive integer value of i, the above integral can expressed in terms of well known special functions. For example the first moment can be written as

$$
\begin{aligned}
v_{y,1}(1) &= \frac{\sqrt{\pi}}{\Gamma\sqrt{2}}\left(\frac{(k_1 - k_2 k_3)}{\sqrt{k_3^2+1}} \text{ erfc}\left(\frac{\alpha\sqrt{k_3^2+1}}{\sqrt{2}}\right)\right. \\
&+ \left. e^{-\frac{\alpha^2}{2}} k_2 \text{ erfc}\left(\frac{k_3\alpha}{\sqrt{2}}\right)\right) + \frac{\mu\mathrm{y}}{\Gamma} T(\alpha, k_3\alpha)
\end{aligned}
$$

where, erfc$(x) = 1 - \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt$ is the complementary error function and $T(x, y)$ is the Owen's T-function [15], given by

$$T(x,y) = \frac{1}{\pi}\int_0^y \frac{e^{-\frac{1}{2}x^2(1+t^2)}}{1+t^2} dt.$$

The special functions erfc(x) and $T(x, y)$ are commonly encountered while integrating univariate and bivariate normal distributions, respectively. Precise numerical tables or accurate closed form analytical approximations exist for both erfc(x) and $T(x, y)$. As the $T(x, y)$ function was not readily available for ease of implementation we used numerical integration to evaluate $T(x, y)$; however, for real SSTA implementation either numerical tables or analytical approximations can be used [16]. Thus similar to [8], the moments of the max can be found in a computationally efficiently manner. Likewise, higher moments can also be found by evaluating the integral given in Equation 4 at higher values of i. Using similar manipulations, the moments of max in all seven regions can be computed.

4. NUMERICAL RESULTS

After deriving analytical expressions for the first three moments , we implemented the new max function in C++. Our goal is to eventually use the proposed max operation in an SSTA framework. Therefore, to emulate the actual use of the true statistical max operation in an SSTA framework, we collected a test suite consisting of

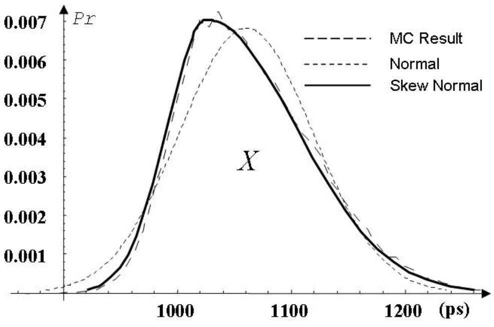

Figure 5: Example: Input X PDF.

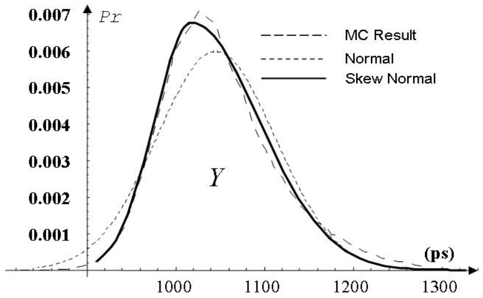

Figure 6: Example: Input Y PDF.

skewed arrival time distributions by running 50,000 Monte Carlo simulations on a small circuit. For each max operation performed, the two input operand arrival time distributions and their resulting maximum arrival time distributions were recorded. For comparison purposes, we also implemented the 5-parameter Clark's max function. The error in both the proposed max operation and Clark's max operation is computed relative to the Monte-Carlo results of the output arrival distribution for each test case.

Now for every test case, we computed the statistical parameters of the two input arrival time distributions, namely, μx_γ, σx_γ, $Sk x_\gamma$, μy_γ, σy_γ, $Sk y_\gamma$ and ρ. These 7 statistical parameters were the input to the proposed skewed normal function. Using the moment matching method presented in Section 2, we first map these parameters to a bivariate skewed normal distribution and then compute the output moments of $\max(X, Y)$ using the analytical results derived in the previous section. An example illustrating the efficacy of the max operation is given in Figure 5, 6 and 7. Given the statistics of $X(1060.55, 58.56, 0.56)$, $Y(1045.53, 66.73, 0.80)$ and their correlation, first the parameters of skewed normal probability distribution function are computed. It can be seen from these figures that the skewed normal distribution accurately represents the Monte Carlo generated skewed arrival time distribution as compared to the symmetric normal for both the inputs. Consequently, as shown in Figure 7 a skewness based treatment of the input arrival time distribution gives $\max(X, Y)$ distribution that accurately matches the MC simulation results.

We found that the error in the max operation based on a normal assumption increases significantly with increase in skewness of the two input arrival time distributions. This is illustrated in Figure 8 where, we show a plot of percentage error in computing the standard deviation of the $\max(X, Y)$ as a function of the skewness in X, Sk. The error in mean was found to be less 1 % in both the

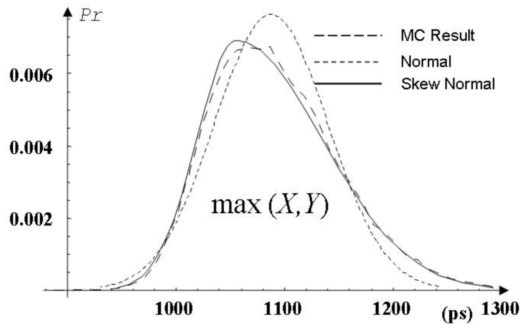

Figure 7: Example: Result $\max(X, Y)$ PDF.

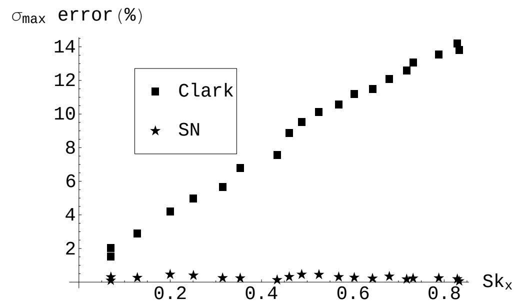

Figure 8: Comparison of standard deviation σ_{max} error (%) as a function of input arrival time skewness Skx

cases. It is evident from this plot that the proposed skewed normal max operation can significantly improve the accuracy of existing SSTA approaches.

Furthermore, as mentioned in [8], the error of the max operation also increases when the difference between μx_γ and μy_γ decreases. We observed a similar trend in our simulation results. In Figure 9 we present error plots of percentage error in standard deviation of output arrival time as a function of $\frac{\mu x_\gamma - \mu y_\gamma}{a}$. It is clear from this plot that the proposed method exhibits much better robustness to difference in the mean of input arrival time distribution.

5. CONCLUSION

In this work we derived novel analytical results for computing

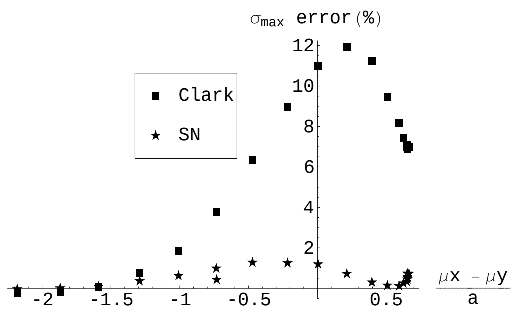

Figure 9: Comparison of standard deviation σ_{max} error (%) as a function of $\frac{\mu \mathbf{x}_\gamma - \mu \mathbf{y}_\gamma}{a}$

the statistical max of two arrival time distributions operation. Using moment based matching given the mean, variance, skewness and correlation of two input arrival time distribution we show how the mean, variance and skewness of the max function can be computed. Thus the proposed skewness based proposed max function can be used to extend existing SSTA framework to propagate the first three moments. A statistical analysis tool that can accurately compute the asymmetry in the circuits arrival time distribution can prove to be extremely useful in statistical optimization. Our numerical results show that the proposed max operation can significantly improve the accuracy of existing SSTA approaches.

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